Rules to Apply Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

- The original integral CANNOT be evaluated by a normal *u*-substitution alone.
- Begin by rewriting the original function as the product of two pieces, *u* and *dv*.
- We must be able to integrate *dv*! e.g., must be able to eval. v from our choice of dv!
- The new integral should be easier than the original problem. If not, try a different choice for *u* and *dv*.

Order in which to choose *u*

Choose *u* according to the *ILATE* rule:

u

- **I** – Inverse Functions $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$
- **L** – Logarithmic Functions $\ln(x), \log(x), \log_b(x)$ for $b > 0$
- **A** – Algebraic Expressions (polynomials, rational functions, etc.) $1, x, x^2$
- **T** – Trigonometric Functions $\sin(x), \cos(x), \tan(x)$
- **E** – Exponential Functions $e^x, e^{-2x}, 3^x$

Tip: In the event of a "tie" in the *ILATE* rule, pick u to be the simplest of the two functions.

useful

Example 3: Evaluate the integral: $\int \underbrace{(\ln x)^2}_{u} \underbrace{dx}_{dv} = I$, apply the IBP method

$\int u\,dv = uv - \int v\,du$

$\int \frac{dt}{t} = \ln|t| + C$

ILATE

$\to (\ln x)^2 = u$

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2 \ln x \, dx}{x} \qquad v = x$

$$I = x(\ln x)^2 - 2\underbrace{\boxed{\int \underbrace{\ln x}_{u} \cdot \underbrace{dx}_{dv}}}_{}\ I_2$$

ILATE
$\uparrow \ln x$

To evaluate I_2, apply IBP again!

$u = \ln x$ $\qquad dv = dx$

$du = \frac{dx}{x}$ $\qquad v = x$

$$I_2 = x \cdot \ln x - \underbrace{\int dx}_{-x} = x \cdot \ln x - x + C_1$$

So, in total: $I = \int (\ln x)^2 \, dx$

$$I = x(\ln x)^2 - 2x\cdot(\ln x) + 2x + C$$

You will need to apply IBP 2X to evaluate

$$\int x^2 e^x \, dx$$

What should we choose for the value of u in the integral

ILATE

$\times \quad \times \quad \times$

e^s

$\sin(s)$

$I = \int \sin[\ln(x)]dx$?

A. $\boxed{\sin(x)}$

B. $\ln(x)$

C. $\sin[\ln(x)]$

D. dx

let's do a s-sub first:

$s = \ln x, \quad ds = \frac{dx}{x}$

$e^s = x$

$\Longleftrightarrow x \cdot ds = dx$

$e^s ds = dx$

(*) $I = \int e^s \cdot \sin(s) ds$

Example 4:

$\int u\,dv$

$= uv - \int v\,du$

Evaluate the integral: $\int \sin[\ln(x)]dx. = I = \int e^s \cdot \underbrace{\sin(s)}_{u}\,\underbrace{e^s\,ds}_{dv}$

apply IBP:

$u = \sin(s)$ $\qquad$ $dv = e^s\,ds$

$du = \cos(s)\,ds$ $\qquad$ $v = e^s$

$\rightarrow I = \sin(s)e^s - \underbrace{\int e^s \cos(s)\,ds}_{I_2}$

Apply IBP a second time!

To evaluate I_2 by parts:

× × × ↗ $\cos(s)$
ILATE
↳ e^s

$$u = \cos(s) \qquad dv = e^s\,ds$$

$$du = -\sin(s)\,ds \qquad v = e^s$$

$$I_2 = e^s\cos(s) + \underbrace{\boxed{\int e^s \sin(s)\,ds}}_{I} + C_1$$

$$I = e^s\sin(s) - e^s\cos(s) - I + C_1 \rightarrow \text{solve for } I$$

$$2I = e^s(\sin(s) - \cos(s)) + C_1$$

$$I = \frac{1}{2}e^s(\sin(s) - \cos(s)) + C$$

we had a good question about using a convention other than the ILATE rule to pick u: (overall, be consistent, try to use ILATE when you can)

$$I = \int e^s \sin(s)\,ds$$

$u = e^s$ $\qquad$ $dv = \sin(s)\,ds$

$du = e^s\,ds$ $\qquad$ $v = -\cos(s)$

$$I = -e^s\cos(s) + \underbrace{\int e^s \cos(s)\,ds}_{I_2}$$

evaluate I_2:

$u = e^s \qquad dv = \cos(s)\,ds$

$du = e^s\,ds \qquad v = \sin(s)$

$$I_2 = e^s \sin(s) - \int e^s \sin(s)\,ds$$

→ leads to the same thing:

$$I = e^s(\sin s - \cos s) - I + C$$

What should we choose for the value of *u* in the integral

$$I = \int e^{2x} \sin(3x)\,dx\,?$$

ILATE

x x x

$\sin(3x)$

e^{2x}

A. $\boxed{\sin(3x)}$

B. e^{2x}

C. $e^{2x}\sin(3x)$

D. dx

Example 5:

Evaluate the integral: $\int e^{2x}\sin(3x)dx.$

Key Idea:
We will do IBP twice and then solve a system for the original integral

Work this example out as an exercise

$$I = \frac{e^{2x}}{13}\left[2\sin(3x) - 3\cos(3x)\right] + C$$

Example 6:

Evaluate the integral: $\int x^4 \ln(x)dx = I$

$\int u\,dv = uv - \int v\,du$

Try to use IBP:

$\overset{\times}{I}\ \overset{\checkmark}{L}\ A\ T\ E$

$\ln(x)$ $\quad x^4$

$u = \ln(x)$ $\quad dv = x^4 dx$

$du = \frac{dx}{x}$ $\quad v = \frac{x^5}{5}$

$I = \frac{x^5}{5}\ln(x) - \frac{1}{5}\int x^4 dx$

So, $I = \frac{x^5}{5}\ln(x) - \frac{1}{25}x^5 + C$

Other examples of the IBP method to try:

Practice which functions to take as u and dv (u-sub first?):

$$I_1 = \int u^5 e^{u^3}\, du$$

$$I_2 = \int x\sqrt{x+1}\, dx$$

$$I_3 = \int x^7 \sqrt{3x^4+5}\, dx$$

$$I_4 = \int x^3 \cos(x^2)\, dx$$

$$I_5 = \int x \sec^2(x)\, dx$$

s-sub: $s = u^3$, $ds = 3u^2 du$

$u^5 du = \frac{1}{3} s\, ds$

$$\to \underbrace{I_1 = \frac{1}{3}\int s e^s\, ds}$$

Then IBP:

I ×

L ×

A ✓ s

T

E

$u = s$

$I_2 = \int x\sqrt{x+1}\,dx$ (have done before with u-sub: $u = x+1$)

what if we wanted to apply IBP?

$u = x$ $\qquad$ $dv = (x+1)^{1/2}\,dx$

$du = dx$ $\qquad$ $v = \frac{2}{3}(x+1)^{3/2}$

$$I_2 = \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\int (x+1)^{3/2}\,dx$$

$$= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(x+1)^{5/2} + C$$

$$I_3 = \int x^7 \sqrt{3x^4 + 5}\,dx$$

$$I_4 = \int x^3 \cos(x^2)\,dx$$

$$I_5 = \int x \sec^2(x)\,dx$$

How to evaluate I_3?

u-sub: $u = 3x^4$

$du = 12x^3\,dx$

$$\underbrace{x^7\,dx}_{\text{figure out what to subst. for } x^7 dx} = \frac{u}{36} \cdot du$$

figure out what to subst. for $x^7\,dx$

$$\rightarrow I_3 = \frac{1}{36} \int u\sqrt{u+5}\,du$$

Then do another sub, or IBP (as above)

$I_4 = \int x^3 \cos(x^2)\,dx$

$I_5 = \int x \sec^2(x)\,dx$

How to evaluate I_4?

→ try a u-sub first

→ $u = x^2$, $du = 2x\,dx$

$$x^3\,dx = \frac{u}{2}\,du$$

$$I_4 = \frac{1}{2}\int u \cdot \cos(u)\,du$$

→ eval by IBP:

"u" $= u$, $dv = \cos(u)\,du$

$(\ldots)$

$\cos u$

ILATE

× × ↑

u

$I_5 = \int x \sec^2(x)dx$

How to evaluate I_5?

Try by parts (IBP):

$u = x$ $\qquad$ $dv = \sec^2 x \, dx$

$du = dx$ $\qquad$ $v = \tan x$

$$I_5 = x\tan x - \int \tan x \cdot dx$$

we know this antiderivative from the examples on u-subs last week

$$= x\tan x - \ln|\sec x| + C$$

$$= x\tan x + \ln|\cos x| + C$$

Any remaining Q's about IBP? ✓

Math 1552

Review of Week 2

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Review Question: Which integrals can we evaluate *by parts*?

$$\cancel{(A)} \int \frac{x^2}{1+x^3}\,dx \to \text{works w/ a u-sub } u = x^3$$

$$\cancel{(B)} \int \frac{1}{x} e^{\ln x}\,dx = \int \frac{x}{x}\,dx = \int dx = x + C$$

$$\checkmark (C) \int x^5 e^{x^3}\,dx \to \text{YES! u-sub first } (u = x^3), \text{ then IBP (see last slides)}$$

$$\checkmark (D) \int x \tan^{-1}(x)\,dx = \underline{I} \text{ (see below)}$$

ILATE

$\uparrow$

$\tan^{-1}(x)$

(D) $u = \tan^{-1}(x)$ $\qquad dv = x\,dx$

$$du = \frac{dx}{1+x^2} \qquad v = \frac{x^2}{2}$$

$$I = \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2\,dx}{1+x^2}$$

<u>write</u>: $\dfrac{x^2}{1+x^2} = \dfrac{x^2+1-1}{1+x^2} = 1 - \dfrac{1}{1+x^2}$

$$I = \frac{x^2}{2}\tan^{-1}(x) - \underbrace{\frac{1}{2}\int dx}_{\frac{x}{2}+C_1} + \underbrace{\frac{1}{2}\int \frac{dx}{1+x^2}}_{\frac{1}{2}\tan^{-1}(x)+C_2}$$

Math 1552

Section 8.3: Powers and Products of Trigonometric Functions

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Today's Goal:

- Use trigonometric formulas to reduce more difficult integrals until we can perform a *u*-substitution.

- Idea: rewrite the function in terms of just one trig function after "breaking off" its derivative for a *u*-substitution

Useful Trig Identities

$$(*)\sin^2 x+\cos^2 x=1$$

if you remember this

$$\tan^2 x + 1 = \sec^2 x \quad (1)$$

$$1+\cot^2 x = \csc^2 x \quad (2)$$

$$(*)1+\tan^2 x=\sec^2 x$$

(Where do these come from?)

$$(*)\sin^2 x=\frac{1}{2}[1-\cos(2x)]$$

$$(1)\ (\sin^2 x+\cos^2 x=1) * \frac{1}{\cos^2 x}$$

$$(2)\ * \frac{1}{\sin^2 x}$$

$$(*)\cos^2 x=\frac{1}{2}[1+\cos(2x)]$$

seen this one before

$$(*)\sin(2x)=2\sin x\cos x$$

$$\sin x\cos y=\frac{1}{2}[\sin(x-y)+\sin(x+y)]$$

$$\sin x\sin y=\frac{1}{2}[\cos(x-y)-\cos(x+y)]$$

Special cases: x=at, y=bt

$$\cos x\cos y=\frac{1}{2}[\cos(x-y)+\cos(x+y)]$$